On the vertical transport due to fingers in double diffusive convection

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Salt fingering across a density interface produced by sugar and salt is investigated experimentally for varying values of initial concentrations and for times sufficiently long to observe the régimes of growth, equilibrium and final decay. Simultaneous measurements of sugar and salt concentrations and interface geometry are made. Calculations based on Stern's collective instability model allow explicit evaluation of the coefficients of the $\frac{4}{3}$-law for the sugar flux in the form suggested by Stern & Turner (1969) for the equilibrium régime. The results are verified by the experimental data. Predictions of the finger wavelengths from this model are in general agreement with the results of previous authors, while the flux-law coefficient is an order of magnitude smaller. It is concluded that the collective instability mechanism is an adequate physical description of the salt-finger mechanism.

1. Introduction

A fluid system vertically stratified by any two diffusing substances can be inherently unstable even when the density decreases upwards (Stommel, Arons & Blanchard 1956; Stern 1960; Turner & Stommel 1964; Walin 1964). If the faster diffusing substance (T) has a gravitationally stable distribution and the slower diffusing substance (S) has a gravitationally unstable distribution, then a necessary condition for instability to infinitesimal perturbations is

$$\frac{K_S}{K_T} < \frac{|\beta\,\partial S/\partial z|}{|\alpha\,\partial T/\partial z|} < 1, \tag{1.1}$$

where $|\beta\,\partial S/\partial z|$ and $|\alpha\,\partial T/\partial z|$ are the magnitudes of the vertical density gradients due to S and T respectively, K_S and K_T are their respective diffusion coefficients, $\alpha = \rho_0^{-1}\,\partial\rho/\partial T$ and $\beta = \rho_0^{-1}\,\partial\rho/\partial S$, where ρ_0 is some constant reference density of the fluid. The resulting motion is the diffusion-driven 'salt-finger' phenomenon described by Turner (1967).

Stern (1969) presented an equilibrium model of the salt-finger structure and examined its stability to a larger scale internal gravity wave perturbation. His analysis was limited to the case in which the diffusing 'substances' are heat (T) and salt (S), and since $K_S/K_T \approx 10^{-2}$ in this case, he assumed $K_S = 0$, essentially neglecting the S-balance equation. Subsequent laboratory experiments (Stern & Turner 1969; Shirtcliffe & Turner 1970) showed that difficult heat control

problems and lateral fluxes of heat could be eliminated by using sucrose (S) and NaCl (T) as the diffusing substances. In this case, $K_S/K_T \approx \frac{1}{3}$ and the S-balance equation cannot be neglected.

The present paper describes a generalization of Stern's model to the latter case in which K_S is not neglected. The next section presents a summary of the theoretical ideas and an explicit derivation of the '$\frac{4}{3}$-law' for the S-flux in the form suggested by Stern & Turner. Subsequent sections describe some experiments using sugar and salt, presenting measurements which support the theoretical ideas.

2. Theoretical considerations

The equilibrium model

We consider a viscous incompressible fluid in a region characterized by vertical gradients of S and T, under the condition (1.1). Cartesian co-ordinates (x, y, z) are employed, with z positive upward. It is convenient to cast the equations in terms of the density fields αT and βS in order to render S and T commensurate. The equations below are written in the form for two solutes, but may be changed to the heat–salt system simply by changing the sign of α.

Invoking the Boussinesq approximation, the equations of motion for perturbations in a system characterized by vertical velocities only (w positive upward) are

$$\nu \nabla_2^2 w = g \, \Delta\rho/\rho_0, \quad K_T \nabla_2^2(\alpha T) = -w\overline{T}_z, \qquad (2.1\,a,b)$$

$$K_S \nabla_2^2(\beta S) = w\overline{S}_z, \quad \Delta\rho/\rho_0 = \alpha T + \beta S, \qquad (2.1\,c,d)$$

where $\overline{S}_z = |\beta \, \partial S/\partial z|$ and $\overline{T}_z = |\alpha \, \partial T/\partial z|$ are assumed constant, ∇_2^2 is the horizontal Laplacian and ν is the kinematic viscosity. The solutions of (2.1) can be written as

$$w = w_0 \sin(x/L) \sin(y/L), \qquad (2 \cdot 2a)$$

$$\alpha T = \alpha T_0 \sin(x/L) \sin(y/L), \qquad (2.2b)$$

$$\beta S = -\beta S_0 \sin(x/L) \sin(y/L), \qquad (2.2c)$$

where L is the horizontal wavelength associated with the fingers. This is the wave form when the finger cells are assumed square in horizontal cross-section, a good approximation in view of the results of Shirtcliffe & Turner (1970). Substitution of (2.2) into (2.1) leads to the following relations:

$$L = (g\overline{S}_z/4\overline{K}_S\nu)^{-\frac{1}{4}} (1 - K_S\overline{T}_z/K_T\overline{S}_z)^{-\frac{1}{4}}, \qquad (2.3)$$

$$w_0 = (g/2\nu)(\beta S_0) L^2 (1 - K_S\overline{T}_z/K_T\overline{S}_z), \qquad (2.4)$$

$$\alpha T_0/\beta S_0 = K_S\overline{T}_z/K_T\overline{S}_z. \qquad (2.5)$$

The vertical flux of S (associated vertical buoyancy flux in cm/s) is

$$\beta F_S = L^{-2} \int_0^L \int_0^L w\beta S \, dx \, dy, \qquad (2.6a)$$

or $\quad \beta F_S = (g/8\nu)(\beta S_0)^2 L^2 (1 - K_S\overline{T}_z/K_T\overline{S}_z). \qquad (2.6b)$

αF_T, the vertical buoyancy flux of T, may be calculated in a similar way, with the result that the ratio of T-flux to S-flux is predicted to be

$$\alpha F_T / \beta F_S = K_S \overline{T}_z / K_T \overline{S}_z. \tag{2.7}$$

It must be emphasized that the above model is a two-dimensional solution of the steady-state Boussinesq equations of motion and may be expected to apply only when conditions are very near critical for the onset of fingers. Since we are considering a system unlimited in vertical extent, the restriction to near-critical conditions may be expressed as $\epsilon \to 0$, where $\epsilon > 0$ is defined by the relation

$$\overline{S}_z / \overline{T}_z = (K_S / K_T)(1 + \epsilon). \tag{2.8}$$

Physically, as ϵ increases, the system becomes more and more supercritical with nonlinear and non-equilibrium effects becoming important. It is reasonable to expect (Snyder 1969; Snyder & Lambert 1966) that in the finite amplitude régimes linearized predictions of wavelengths and wave forms will agree fairly well with observations, while the amplitudes will be significantly modified by nonlinear effects. On this basis, (2.3), (2.4) and (2.6b) should be correct in *form*, but the amplitudes must be determined by experiment and/or further refinements using nonlinear theory. The experiments described subsequently must be considered highly supercritical with respect to the formation of fingers.

Collective instability

Using the one-dimensional form of the more general equilibrium model described in the preceding section, it is possible to consider the stability of a system of fingers using the same procedure as Stern (1969). The mechanism of instability is pictured as a large-scale (with respect to L) internal wave, which rotates and translates groups of fingers collectively through a shearing action. It is assumed that the internal wave does not alter the magnitude of the flux, allowing the use of (2.6). Under these assumptions, and limiting the calculation to one horizontal dimension, a repetition of Stern's calculation for finite K_S leads to the condition

$$(\beta F_S - \alpha F_T) / \nu(\overline{T}_z - \overline{S}_z) = A \tag{2.9}$$

for marginal stability, where A is some critical dimensionless number of order 1. βF_S is the magnitude of the downward buoyancy flux of S and αF_T is the magnitude of the upward flux of T, so that $\beta F_S - \alpha F_T$ is the net vertical buoyancy (density) flux. Equation (2.9) indicates that the fingers will break down when the net unstable buoyancy flux carried by them is greater than or equal to some particular fraction of the net density gradient through the finger region, scaled in terms of the kinematic viscosity ν. This may be compared with Stern's (1969) result (his equation (4.4)), in which a balance between S-flux and T-gradient is obtained, the gradient of S having been neglected.

The addition of a second horizontal dimension does not alter the physics of the problem, and (2.9) may be expected on purely dimensional grounds. We have chosen to leave the magnitude of A unspecified, since it is not determined explicitly either by Stern or by our calculation, and it will be determined experimentally. The number A contains both a critical angle of rotation of the group

of salt fingers and the expression $g\bar{T}_z L^4/K_T \nu$, which Stern took to be 1. Equation (2.3) shows that this quantity may vary by at least two orders of magnitude, depending on whether one looks at a heat–salt system or a two-solute system. Further, the assumptions used to obtain (2.9) obscure its value even more, but the physical result of a critical ratio of buoyancy flux to density gradient remains. The experiments will show that, at least for the two-solute case, A is of order 10^{-3}.

Vertical flux laws

We now wish to apply these results to the finger interface which forms when a layer of fluid of density $1 + (\beta\Delta S)_0$ is placed over a layer of fluid of density $1 + (\alpha\Delta T)_0$, such that

$$K_S/K_T < (\beta\Delta S)_0/(\alpha\Delta T)_0 < 1. \tag{2.10}$$

It is well known (Turner 1967; Stern & Turner 1969; Shirtcliffe & Turner 1970) and can be observed that, in such an interface, the fingers are sheared off by a larger scale convection above and below, which limits the vertical extent of the finger region to a thickness h, without destroying it completely. The finger region remains for a period of time which depends on the initial values $(\alpha\Delta T)_0$ and $(\beta\Delta S)_0$ and then gradually thickens as S is transported downwards and T upwards, decreasing $\alpha\Delta T$ and $\beta\Delta S$.

If the interface is pictured as a region of stable fingers, and the convecting region as one in which the fingers have become unstable, then it is a good approximation to consider the S-flux carried by the fingers in the interface region to be given by (2.6). In the convecting region the fingers have become unstable. In that region, or, more rigorously, at the boundary between that region and the finger region, it is reasonable to assume that $(\beta F_S - \alpha F_T)/\nu(\bar{T}_z - \bar{S}_z)$ is very close to the critical value of A, since the fingers are very close to the condition of marginal stability. By virtue of continuity, the flux carried by the fingers must equal the flux carried through the boundary, and we may thus equate the two flux laws (2.6) and (2.9). We now write (2.9) as

$$\beta F_S(1 - \gamma) = \nu A(\bar{T}_z - \bar{S}_z), \tag{2.11}$$

where $\gamma = \alpha F_T/\beta F_S$ is the empirical ratio of T-flux to S-flux through the interface. In the equilibrium régime, γ is determined experimentally to be constant, contrary to the prediction of (2.7); the implications of this fact will be discussed later. In the sugar–salt experiments, it is reasonable to approximate the gradients in the finger region by $\bar{T}_z = \alpha\Delta T/h$ and $\bar{S}_z = \beta\Delta S/h$ (see figure 3 below).

Equations (2.6), (2.3) and (2.11) may now be combined to give

$$\beta S_0 = \eta(\beta\Delta S), \tag{2.12}$$

where

$$\eta = [2A(\nu/K_S)(N-1)/(1-\gamma)]^{\frac{1}{2}}(L/h) \tag{2.13}$$

and $N = \alpha\Delta T/\beta\Delta S$. It can be seen that the amplitude of the S perturbation varies and adjusts itself according to the density ratio N, the Schmidt number ν/K_S and the aspect ratio L/h, or the ratio of finger width to finger length. Equations (2.13), (2.3) and (2.6) may now be combined to give a flux law which agrees in form with equation (1) of the paper by Stern & Turner:

$$\beta F_S = C(\beta\Delta S)^{\frac{4}{3}}, \tag{2.14a}$$

Run	$(\alpha\Delta T)_0$	$(\beta\Delta S)_0$	$\nu\times10^2$	$(g\nu^2/4K_S)^{\frac{1}{3}}$	η
9 June	0·100	0·090	1·53	22·6	0·126
11 June	0·100	0·064	1·30	20·2	0·082
24 June	0·100	0·080	1·45	21·8	0·117
9 July	0·096	0·070	1·38	21·0	0·087

TABLE 1. Summary of initial conditions and experimental parameters for each run. Other values used are $g = 0.98\times10^3$ cm/s^2; $K_S = 0.5\times10^{-5}$ cm^2/s; $K_T = 1.5\times10^{-5}$ cm^2/s

where
$$C = (g\nu)^{\frac{1}{3}}f, \tag{2.14b}$$

$$f = 4^{-\frac{1}{3}}A(N-1)(1-\gamma)^{-1}(\nu/K_S)^{\frac{1}{3}}(1-\mu N)^{\frac{1}{3}}(L/h)^{\frac{4}{3}} \tag{2.14c}$$

and $\mu = K_S/K_T$. Equation (2.14a) is the vertical flux law for a finger interface of thickness h, separating two convecting regions when the magnitudes of the differences in density across the interface due to S and T are $\beta\Delta S$ and $\alpha\Delta T$ respectively, and L is the horizontal width of the fingers. The expression C will be constant only if the sources of S and T are just sufficient to maintain the flux, so that $\alpha\Delta T$, $\beta\Delta S$ and L/h do not vary in time. If the system is running down, N will vary and L/h will adjust itself such that (2.15) remains valid. In the sugar–salt experiments C is a very slowly varying function of time.

3. Experiments

Experiments were carried out at room temperature in a Plexiglas tank 26 cm long, 10 cm wide and 25 cm deep. Concentrations used resulted in density differences sufficiently large that small variations in room temperature (2–3 °C) could be safely neglected. The general procedure was to prepare separate solutions of sugar and salt at the desired densities and allow them to stand overnight to establish them at room temperature. The amounts of sugar and salt added raised the initial densities by amounts $(\beta\Delta S)_0$ and $(\alpha\Delta T)_0$ respectively; the specific gravities were measured using hydrometers accurate to within ±0.001. The density of water at room temperature was found to be 1·000 within the accuracy of the hydrometers. An approximate initial value of $(\alpha\Delta T)_0 = 0.10$ was used in each experiment, and thus the maximum uncertainty in both $(\alpha\Delta T)_0$ and $(\beta\Delta S)_0$ was about 2 %. $(\beta\Delta S)_0$ was varied from run to run within the confines of the salt-finger criteria (1.1) and (2.10). Initial values and other parameters are summarized in table 1.

The salt solution was placed in the tank to a depth of 12 cm. The sugar solution was then placed carefully over the salt solution using a styrofoam float wrapped with cheese-cloth which allowed the sugar solution to flow in with a minimum of disturbance at the interface. 12 cm of sugar solution were added in this way, with total filling times averaging 15 to 20 minutes. When the tank was full, the filler float was removed and the tank covered to minimize evaporative effects. The apparatus with an interface formed is shown schematically in figure 1, and may be compared with figure 1 of Shirtcliffe & Turner (1970), which is an actual photograph of such an interface.
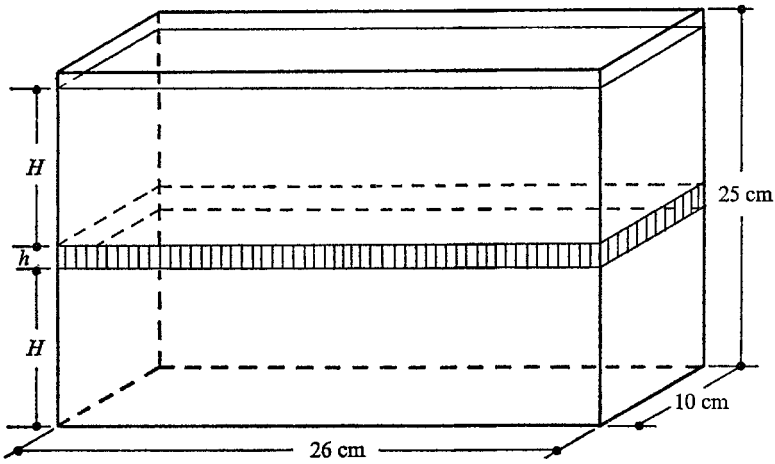
FIGURE 1. Schematic diagram of experimental tank showing a two-layer experiment with a finger interface of thickness h separating two well-mixed convecting layers of thickness H.
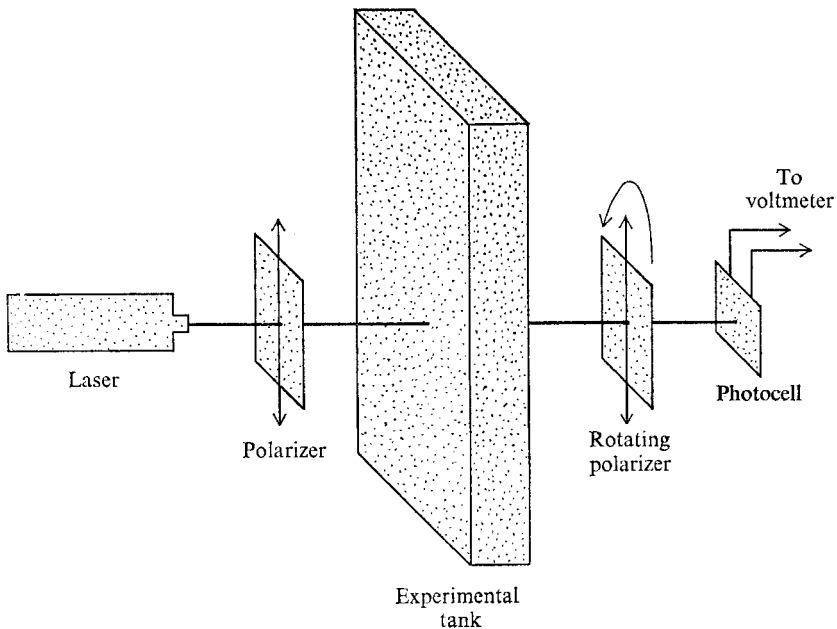

FIGURE 2. Schematic diagram of apparatus (side view) showing the polarimeter used to measure the vertical profile of sugar concentration.

The development of the convection was observed optically using the shadow-graph technique described by Shirtcliffe & Turner (1970). Both the stationary interface and the strong convection above and below could be observed in this manner, and the thickness of the interface, h, was determined by judging where the horizontal motion in the turbulent convection came to rest at the upper and lower edges of the finger region.

Measurements of the horizontal-average sugar concentration were made using a polarized laser beam and a rotating analyser as shown in figure 2. Since sugar
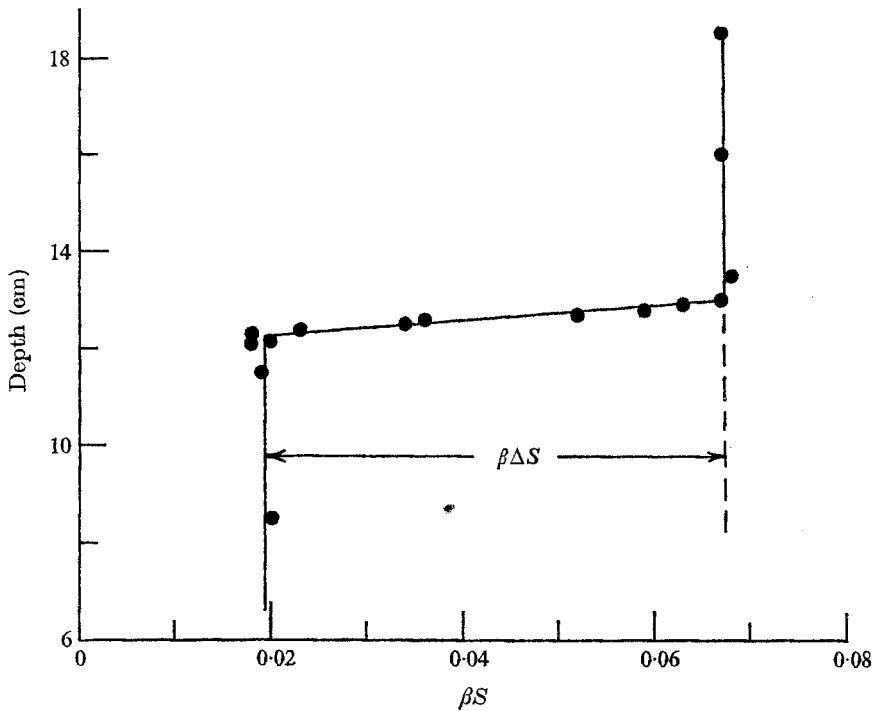
FIGURE 3. Vertical profile of sugar concentration βS as a function of depth at $t = 70$ min in experiment 3, June 1970. $\beta \Delta S$ is the density contrast due to sugar across the interface and each point represents an individual measurement.

is optically active, the plane of polarization of the beam is rotated through an angle which is proportional to the concentration of sugar (Jenkins & White 1957). By moving the laser beam up and down, horizontally averaged vertical profiles of βS could be obtained, and an example of such a profile is shown in figure 3. The vertical resolution is estimated to be about 1 mm. $\beta \Delta S$ ($\alpha \Delta T$) is defined as the difference in sugar (salt) concentration between the upper layer and lower layer, as shown for sugar in figure 3.

The strong convection above and below the interface kept both deep layers quite well-mixed during the experiments, as is indicated by the measurements shown in figure 3. It can be reasonably assumed that T was as uniformly distributed in these layers as S. The concentration of T was then determined by measuring the indices of refraction of both top and bottom layers as functions of time. Knowing the index of refraction and the sugar concentration simultaneously allowed determination of the T-concentration from graphs such as those of figure 18 of Shirtcliffe & Turner (1970). The index of refraction was determined by leaching a small sample from a mixed layer using a hypodermic syringe inserted well into the tank through a resealable rubber grommet located on the side of the tank. Several such grommets were located at various depths to ensure that a sample from the mixed layer was obtained. The index of refraction n was measured for each sample using an American Optical pocket refractometer. In general, three individual readings at each point in time were

averaged together to give a single data point. Each value of n was then converted to the corresponding value αT as indicated above. αT was then plotted as a function of time for both top and bottom layers, and a smooth curve was fitted to these points to give $\alpha T(t)$ for each layer. At any instant of time, the difference between these two curves gave $\alpha \Delta T$ at that time.

Using the polarization technique, βS was determined for each layer to within 3 %, and $\beta \Delta S$ was calculated to within about 5 %, taking into consideration both instrument sensitivity and observed scatter. $\alpha \Delta T$ was determined to about the same accuracy, and smoothing the measurements as a function of time improved the accuracy by at least a factor of 5. Nevertheless, $\alpha \Delta T$ and $\beta \Delta S$ are quoted as known to within 5 %, and thus the values of N are quoted to within 10 %. The interface thickness h could be measured to within about 5 % initially and to better than 1 % towards the end of the experiments. Plotting h vs. time showed that its growth was almost linear in time, and a smooth curve through the measured values removed much of the scatter due to uncertainties in individual measurements. h could also be determined by noting the extent of the gradient region in profiles such as that in figure 3. Excellent agreement was found between this method and the shadowgraph technique described previously, and the data presented here result from the shadowgraph. Measurements of h will be considered accurate to within 1 % except, perhaps, near the beginning and end of the experiments.

The values of the physical parameters K_S and K_T for dilute solutions were taken from the handbooks and used without correction. $K_S = 0.5 \times 10^{-5} \, \text{cm}^2/\text{s}$ and $K_T = 1.5 \times 10^{-5} \, \text{cm}^2/\text{s}$, and hence $K_S/K_T \approx \frac{1}{3}$. Their uncertainties are not considered here, since it is expected that the solutions used are sufficiently dilute for these values to be valid. The kinematic viscosity, however, varies with sugar concentration and is a source of considerable uncertainty. The values cited in table 1 are the means of the values for the initial solutions of sugar and salt. Use of the mean value is supported by independent measurements of the viscosity of fluid sampled directly from the finger region in earlier experiments. It can be expected, however, that ν will vary significantly from one layer to the next, and possibly even horizontally from one finger to the next, and thus introduce an uncertainty of at least 10 % in the final results. In summary, the quantities h, $\alpha \Delta T$ and $\alpha \Delta S$ were determined as functions of time, zero time being defined as that moment when pouring began. Typical experiments lasted for 6–12 h and measurements were made throughout these times. Four separate runs are reported here and the conditions for all experiments are listed in table 1.

4. Results and discussion

The flux ratio γ is determined empirically by plotting $\alpha \Delta T$ vs. $\beta \Delta S$ as shown in figure 4. The slope is equal to γ, since

$$\gamma = \frac{\alpha F_T}{\beta F_S} = \frac{d(\alpha \Delta T)/dt}{d(\beta \Delta S)/dt} = \frac{d(\alpha \Delta T)}{d(\beta \Delta S)}. \tag{4.1}$$

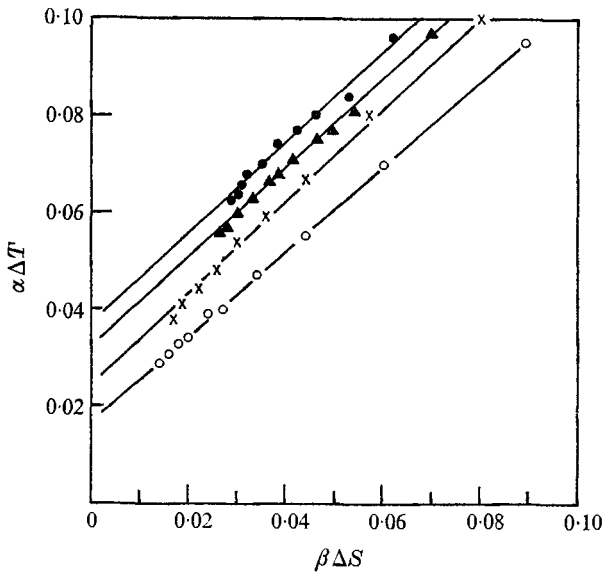Initial values occur at the upper right corner for each curve, and progress in

FIGURE 4. Plot of $\alpha\Delta T$ vs. $\beta\Delta S$. Initial values occur at upper right corner and experiments progress in time towards the lower left corner. ○, 9 June; ●, 11 June; × 24 June; ▲, 9 July.

time towards the lower left corner as sugar is transported downwards and salt upwards. Points on each curve represent data taken over at least 6 h. During the major part of each run it can be seen that γ (the slope) is the same constant for each run. Averaging the measurements from all four runs gives $\gamma = 0.92 \pm 2\%$, which is in good agreement with the value cited by Stern & Turner (1969).

The fact that γ is a constant in this régime and not as given by (2.7) can be partially explained in terms of the collective instability model. As the fingers form, the density flux increases, reaching a critical value given by (2.9) at which the fingers break down, forming a turbulent convection pattern. This process takes place much more slowly when the initial value of N is near μ^{-1}, and can be easily observed by means of the shadowgraph. In the equilibrium state (with the finger interface separating strongly convecting layers) γ remains constant even though βF_S and αF_T individually decrease. Equation (2.7) applies strictly only when there are fingers alone in the absence of strong convection, and, indeed, $\mu N \to \gamma$ as the convection dies out. It appears that if this is the case, as the experiments indicate, then the formation of convecting layers from salt fingers implies a limiting of the flux as indicated in (2.9).

The departures from linearity at the lower left of figure 4 are almost certainly due to the approach of the system to the state in which the convection is dying out, the interface depth is approaching the depth of the tank and the collective instability model no longer applies. As the convection dies out, and the fingers extend throughout the entire region, the equilibrium model should apply throughout the entire region and the step structure disappears.

The next step is to notice that the finger wavelength $(L = \lambda/2\pi)$ may be calculated from measurements of $\alpha\Delta T$, $\beta\Delta S$ and h by means of (2.3). We let
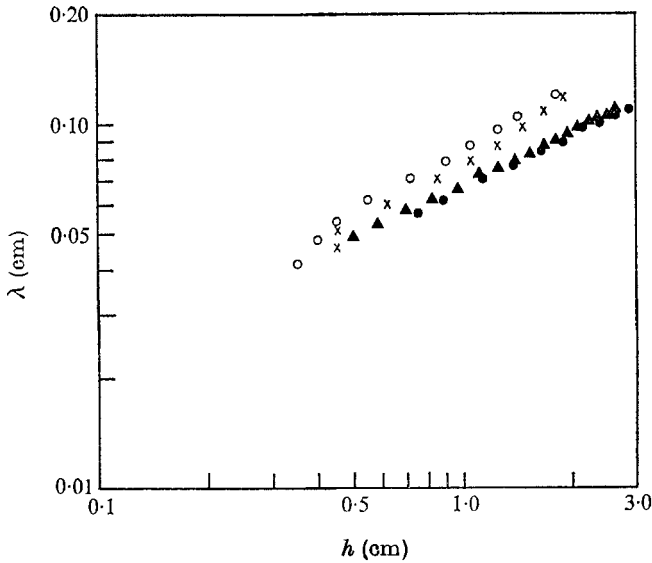
FIGURE 5. Calculated values of the wavelength λ as functions of measured values of the finger length h. $\bigcirc$, 9 June; $\bullet$, 11 June; $\times$, 24 June; $\blacktriangle$, 9 July.

$\bar{S}_z = \beta \Delta S/h$ and $\bar{T}_z = \alpha \Delta T/h$ so that

$$\lambda/h^{\frac{1}{4}} = 2\pi(4K_S\nu/g)^{\frac{1}{4}}[\beta \Delta S(1-\mu N)]^{-\frac{1}{4}}. \tag{4.2}$$

During the course of a single experiment, $\beta \Delta S$ and $(1-\mu N)$ both decrease slowly so that $\lambda/h^{\frac{1}{4}}$ is a slowly increasing function of time. Measurements of $\beta \Delta S$, $\alpha \Delta T$, and h as functions of time permit the calculation of corresponding values of λ to within about 6 %. A plot of λ *vs. h* for all runs considered is shown in figure 5 and is presented in this manner in order to allow comparison with the results of Shirtcliffe & Turner (1970).

If the right-hand side of our equation (4.2) were constant, we could compare its value with that given by the line drawn through their data, which has a slope of $\frac{1}{4}$. The fact that it is not constant probably indicates the degree of validity of the present model. The data in our figure 5 are consistent with the interpretation $\lambda/h^{\frac{1}{4}} = $ constant, but the important point is that they compare very well with those of Shirtcliffe & Turner between $h = 1$ cm and $h = 3$ cm, where both the present model and the data of Shirtcliffe & Turner are expected to be most reliable. In fact, the uncertainties in the photographic data of Shirtcliffe & Turner for $h < 1$ cm and the trend of all the data for $h > 3$ cm seems to imply that the present model provides a reasonable description of what is happening. The agreement is clearly better than one might expect and strengthens the argument that at least a quasi-equilibrium state exists in the experimental model.

The flux law may be tested by assuming that C in (2.14) is constant. If A is constant (which we *will* assume) then this is not strictly true, since C appears to be a slowly varying function of N and L/h. However, values of C/A tabulated in table 2 show that it varies by less than 10 % throughout most of the experiment, or for a period of 5–10 h (discarding the initial value). It is therefore a very

Time (min)	$[\beta\Delta S \times (1-\mu N)]^{\frac{1}{4}}$	h (cm)	λ (cm)	$(1-\mu N)^{\frac{1}{3}} \times (N-1)$	$(\lambda/2\pi h)^{\frac{4}{3}}$	C/A (cm/s)	η
0	0·465	0·45	0·046	0·209	$4·11 \times 10^{-3}$	0·234	0·090
50	0·417	0·45	0·051	0·328	4·72	0·423	0·124
100	0·384	0·62	0·060	0·411	3·83	0·428	0·123
150	0·357	0·85	0·070	0·491	3·08	0·412	0·116
200	0·331	1·05	0·079	0·590	2·74	0·442	0·119
250	0·316	1·25	0·087	0·618	2·46	0·415	0·113
300	0·292	1·46	0·098	0·693	2·36	0·447	0·119
350	0·275	1·66	0·108	0·744	2·27	0·461	0·123
400	0·256	1·87	0·119	0·784	2·17	0·463	0·124

TABLE 2. Example of time development of conditions across the interface: Data for 24 June. $\mu = K_S/K_T = \frac{1}{3}$; and $N = \alpha\Delta T/\beta\Delta S$.
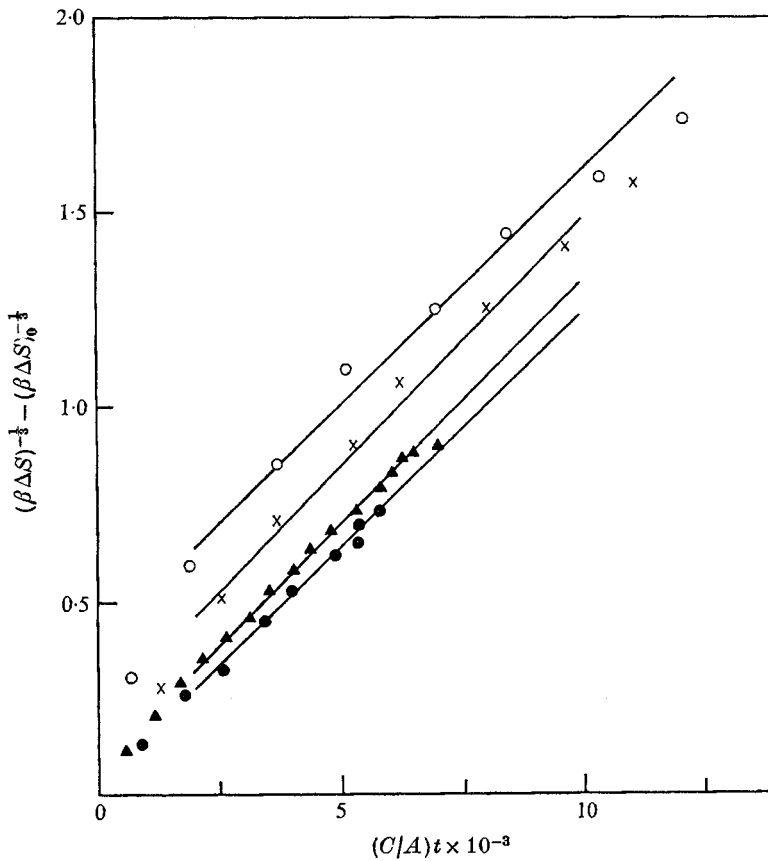

FIGURE 6. Plot of $(\beta\Delta S)^{-\frac{1}{3}} - (\beta\Delta S)_0^{-\frac{1}{3}}$ vs. $(C/A)t \times 10^{-3}$. ○, 9 June; ●, 11 June; ×, 24 June; ▲, 9 July.

good approximation to say that C is constant for periods of at least several minutes, and probably longer. Essentially, the system appears to move from one equilibrium state to another, as was noted by Stern & Turner.

Thus, we may write $d(\beta\Delta S)/dt = -2\beta F_S/H$, where H is the thickness of the deep convecting layers above and below the interface. If C is constant, or slowly varying, this may be immediately integrated to give

$$(\beta\Delta S)^{-\frac{1}{3}} - (\beta\Delta S)_0^{-\frac{1}{3}} = 2Ct/3H = A(2/3H)(C/A)t. \qquad (4.3)$$

H may also be considered constant during the times of interest in these experiments and is taken to be $10\,\mathrm{cm}$. C/A may now be calculated from the measured values of $\beta\Delta S$, $\alpha\Delta T$ and h, and the calculated values of L for each experiment (cf. table 2). We next plot $(\beta\Delta S)^{-\frac{1}{3}} - (\beta\Delta S)_0^{-\frac{1}{3}}$ vs. $(Ct/A) \times 10^{-3}$; the results are shown in figure 6. The data for each run can be treated as straight lines, except near the beginning and the end of each experiment, and the slope must be $2A/3H$. Calculations from these data give $A = 1{\cdot}8 \times 10^{-3}$, with corresponding values of C ranging from $0{\cdot}5 \times 10^{-3}$ to $0{\cdot}75 \times 10^{-3}$. The uncertainty, considering all possible sources, is estimated to be no greater than $10\,\%$.

The primary result of these experiments is the verification of the flux law, equation (2.14). This equation is essentially the '$\frac{4}{3}$-law' suggested by Stern & Turner with an explicit functional form for C. It appears that, as N changes, C also changes, but only gradually, and the horizontal wavelength λ and the thickness of the interface adjust themselves accordingly. The $2\,\%$ uncertainty in γ, however, introduces a $20\,\%$ uncertainty in $1-\gamma$, and it is this quantity which appears in the flux law (2.14). Therefore the final determination of A cannot be certain to better than $20\,\%$ even though the scatter is found to be much less. The fact that it is almost three orders of magnitude smaller than the value predicted by the collective instability model is due primarily to the assumptions leading to that model as described in §2.

On the basis of these results, we may proceed to calculate the S-amplitude βS_0 and hence the vertical velocity amplitude w_0 in the fingers. βS_0, or more precisely the ratio of βS_0 to $\beta\Delta S$, is given by equations (2.12) and (2.13). Although some systematic variations are observed, η appears to be constant for each run to well within $10\,\%$. The values for one run are given in table 2 and the mean values for each run are given in table 1. The conclusion is that the amplitude βS_0 is of the order of $\frac{1}{10}$ of the difference $\beta\Delta S$ and that the actual magnitude depends systematically on the initial concentration, $(\beta\Delta S)_0$. Clearly this is only an estimate, since in reality βS_0 is probably a function of z.

The vertical velocity amplitude w_0 can now be calculated from (2.4), and is found to be constant within $10\,\%$ for all runs, viz., $w_0 = 1{\cdot}30 \times 10^{-3}\,\mathrm{cm/s}$. This value is somewhat more than an order of magnitude greater than the rate of growth of the interface, further indicating the limiting effects of the convection on the interface as a whole and the surprising efficiency of the fingers themselves in transporting S and T vertically. It is somewhat surprising to find so little deviation in this value over such a wide variation of parameters. Perhaps further refinements of the experiment will reveal systematic variations in w_0, but they are not apparent here. Finally, we may estimate the efficiency of transport in

another way, computing a 'Nusselt' number of S-transport by comparing the total flux βF_S with purely diffusive flux $\beta F_S^* = K_S \beta \Delta S/h$. The ratio N_S can then be defined as

$$N_S = F_S/F_S^* = [A(N-1)/(1-\gamma)][g\nu^2\lambda^4(1-N)\beta\Delta S/64\pi^4 K_S^4 h]^{\frac{1}{3}}. \qquad (4.4)$$

For a typical experiment N_S is of the order of 50, indicating that the vertical flux due to fingering is approximately two orders of magnitude greater than it would be if diffusion were the only transport mechanism.

5. Summary and conclusions

The primary discrepancies between these results and those of previous authors are the fact that A is considerably less than the predicted value of order 1 and that the value of C is more than an order of magnitude smaller than that determined by Stern & Turner (1969). The former is probably due to the nature of the assumptions in the collective instability model, which have been discussed previously. It is possible that the latter is due to the difference in procedure between these experiments and those of Stern & Turner. The total time of their experiments was reported to be about 30 min (see figure 3 and table 1 of their paper). In our experiments, although this is not apparent in figure 6, the quasi-equilibrium state was not well established for at least 50–100 min (the first two points for each set in figure 6). If a slope is measured in this early time-dependent régime, a value of A as much as two or three times larger than that cited above can be obtained, and it is likely that measurements between $t = 0$ and $t = 50$ min would give values of A as high as 10^{-2}, which would yield a value of C in reasonable agreement with that of Stern & Turner. In general, the assumption that the finger interface in a two-layer model passes through a succession of quasi-equilibrium states is found to provide an adequate, internally consistent description of salt fingers for the case when the two solutes are sugar and salt. A flux law is derived and the constants are evaluated experimentally. The model is found to break down somewhat near the beginning and end of the experiments, but predicts measured results throughout a wide range of conditions. The data generally apply to the finger interface during the time in which it is well established, with strong larger scale convection above and below. Discrepancies near the beginning and end of the experiments can be attributed to non-equilibrium and time-dependent effects. Some aspects of the initial growth process have been observed qualitatively and photographed extensively (Lambert 1971). Once established, the finger interface transports just enough buoyancy to maintain the larger scale convection while still preserving the finger interface. As time goes on, $\alpha\Delta T$ and $\beta\Delta S$ decrease and the parameters of the interface slowly change, but even for periods of several minutes, the equilibrium model appears to provide an adequate description of the kinematics of the flow. As the large-scale convection dies out, the equilibrium model of fingers becomes quantitatively more applicable.

It is possible that this model may apply to the heat–salt system with perhaps some changes in the values of the constants A and γ and the appropriate physical

parameters, but with the form of the equations remaining the same. Measurements in a heat–salt system are needed to determine these values and thus test the validity of extending the model to describe the oceanic microstructure.

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